Exercise 3

A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.

(a) Find an expression for the number of bacteria after t hours.

(b) Find the number of bacteria after 3 hours.

(c) Find the rate of growth after 3 hours.

(d) When will the population reach 10,000?

Solution

Part (a)

Assume that the rate of population growth is proportional to its size.

$$\frac{dP}{dt} \propto P$$

Change this proportionality to an equation by introducing a constant k.

$$\frac{dP}{dt} = kP$$

Divide both sides by P.

$$\frac{1}{P}\frac{dP}{dt} = k$$

Rewrite the left side using the chain rule.

$$\frac{d}{dt}\ln P = k$$

The function that you take a derivative of to get k is $kt + C$, where C is any constant.

$$\ln P = kt + C$$

Exponentiate both sides to get P.

$$\begin{aligned} P(t) &= e^{kt+C} \\ &= e^C e^{kt} \end{aligned}$$

Use a new constant P_0 for e^C.

$$P(t) = P_0 e^{kt} \tag{1}$$

The bacteria culture initially contains 100 cells.

$$P(0) = P_0 e^{k(0)} = 100 \quad \rightarrow \quad P_0 = 100$$

Equation (1) then becomes

$$P(t) = 100e^{kt}.$$

Use the fact that the population is 420 after 1 hour to determine k.

$$P(1) = 100e^{k(1)} = 420 \quad \rightarrow \quad e^k = 4.2 \quad \rightarrow \quad k = \ln 4.2 \approx 1.43508 \text{ hour}^{-1}$$

Therefore, the bacteria population after t hours is

$$\begin{aligned} P(t) &= 100e^{(\ln 4.2)t} \\ &= 100e^{\ln (4.2)^t} \\ &= 100(4.2)^t. \end{aligned}$$

Part (b)

After 3 hours the bacteria population is

$$P(3) = 100(4.2)^3 \approx 7409 \text{ cells.}$$

Part (c)

After 3 hours the rate of population growth is

$$\left.\frac{dP}{dt}\right|_{t=3} = kP(3) = (\ln 4.2)[100(4.2)^3] \approx 10632.3 \ \frac{\text{cells}}{\text{hour}}.$$

Part (d)

To find when the population will reach 10,000, set $P(t) = 10{,}000$ and solve the equation for t.

$$\begin{gathered} P(t) = 10\,000 \\ 100(4.2)^t = 10\,000 \\ (4.2)^t = 100 \\ \ln(4.2)^t = \ln 100 \\ t \ln 4.2 = \ln 100 \\ t = \frac{\ln 100}{\ln 4.2} \approx 3.20899 \text{ hours} \end{gathered}$$